Some conjectures of Graffiti.pc on total domination

Ermelinda DeLaViña*, Qi Liu**, Ryan Pepper*,
Bill Waller*, and Douglas B. West**,[1]
*University of Houston-Downtown, Houston, TX, 77002
**University of Illinois, Urbana, IL, 61801

In this paper, we present some new upper and lower bounds on the total domination number of a graph that originated as conjectures of Graffiti.pc.
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Introduction and Key Definitions

We limit our discussion to graphs that are simple and finite of order n. Although we often identify a graph G with its set of vertices, in cases where we need to be explicit we write $V(G)$. A set M of vertices of G is said to *dominate* G provided each vertex of G is either in M or adjacent to a vertex of M. The *domination number* of G is the minimum order of a dominating set. A dominating set M of G is said to *totally dominate* G provided each vertex of G is adjacent to a vertex of M. The *total domination number* of G is the minimum order of a totally dominating set. The total domination number is denoted by $\gamma_t = \gamma_t(G)$. The minimum order of a *connected* dominating set is denoted by $\gamma_c = \gamma_c(G)$. Other definitions will be introduced immediately prior to their first appearance.

The total domination number of a graph was first introduced in [3]. This invariant remains of interest to researchers as evidenced by numerous recent papers. Various upper and lower bounds on total domination have been discovered. The domination number has, of course, been well studied ([15], [16]).

Graffiti, a computer program that makes conjectures, was written by S. Fajtlowicz and dates from the mid-1980's. Graffiti.pc was written by E. DeLaViña in 2001. The operation of Graffiti.pc and its similarities to Graffiti are described in [4] and [5]; its conjectures can be found in [6]. A numbered, annotated listing of several hundred of Graffiti's conjectures can be found in

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[10]. Both Graffiti and Graffiti.pc have correctly conjectured a number of new bounds for several well studied graph invariants; bibliographical information on resulting papers can be found in [7].

Recently, DeLaViña used Graffiti.pc to generate conjectures involving the total domination number. Several of the consequent conjectures either follow from known results or have been resolved. A numbered, annotated listing of Graffiti.pc's total domination conjectures and their current status can be found in [6]. In this paper, we present the proofs of several of these conjectures that are new, so far as we can determine, as well as the proofs of some related conjectures motivated by Graffiti.pc's conjectures.

Graffiti.pc employs two main strategies for generating conjectures. The first of these is due to Fajtlowicz and is known as the "Dalmatian heuristic" [11]. All but two of the Graffiti.pc conjectures cited in this paper are Dalmatian conjectures. The other two are Sophie conjectures. The "Sophie heuristic" (due to DeLaViña and B. Waller) is the second main strategy Graffiti.pc uses to generate conjectures. See [8] for a description of Graffiti.pc's Sophie heuristic.

Results and Proofs

The *eccentricity* of a vertex v of a connected graph G is the maximum of the distances from v to the other vertices of G. The maximum eccentricity taken over all vertices of G is called the *diameter* of G and is denoted by $d = d(G)$. The minimum eccentricity taken over all vertices of G is called the *radius* of G and is denoted by $r = r(G)$. The radius of a graph has sometimes been used to provide lower bounds for domination-related invariants. One of the first results along these lines is the following theorem, which originated as a conjecture of the computer program Graffiti [10]. There are several independent proofs of this theorem (see [9], [12], [13], [14]). Let $\alpha = \alpha(G)$ denote the *independence number* of G: this is the maximum order of a set of pairwise non-adjacent vertices of G.

Theorem 1: *Let G be a connected graph. Then*

$$\alpha \geq r.$$

A stronger result due to Fajtlowicz [13] uses the radius to provide a lower bound for the *bipartite number* $b = b(G)$ of a graph G. This is the maximum order of an induced bipartite subgraph of G.

Theorem 2: *Let G be a connected graph. Then*

$$b \geq 2r.$$

We now show that the radius can also provide a lower bound for the total domination number. This is Graffiti.pc's Conjecture 230 in [6].

Theorem 3: *Let G be a connected graph with $n > 1$. Then*

$$\gamma_t \geq r.$$

The proof of this theorem is presented after the proofs of the following lemmas.

Lemma 1: *Let T be a tree with dominating set M. Then the subgraph induced by $T - M$ has at most $k - 1$ edges, where k is the number of components of the subgraph induced by M.*

Proof. Let e_1 denote the number of edges with both endpoints in M, let e_2 denote the number of edges with both endpoints in $T - M$, and let e_3 denote the number of edges with one endpoint in M and one endpoint in $T - M$. Proceeding by contradiction, suppose there are k or more edges induced by $T - M$. Namely we assume:

(1) $$e_2 \geq k.$$

First, since T is a tree,

(2) $$n - 1 = e_1 + e_2 + e_3.$$

Next, since the graph induced by M is a forest with k trees,

(3) $$e_1 = |M| - k.$$

Moreover, because each of the $n - |M|$ vertices of $T - M$ is dominated by a vertex in M,

(4) $$e_3 \geq n - |M|.$$

Finally, we put inequalities (1) and (4) together with equations (2) and (3) to obtain:

$$n - 1 = e_1 + e_2 + e_3 \geq (|M| - k) + k + (n - |M|) = n,$$

a contradiction. Consequently, $e_2 \leq k - 1$ as claimed. ■

A *spanning tree* of a connected graph G is a subgraph that contains all vertices of G and is a tree.

Lemma 2: *Let G be a connected graph with $n > 1$, and let M be a minimum total dominating set of G. Then there exists a spanning tree S of G such that M is a minimum total dominating set of S.*

Proof. If G is a tree, then put $S = G$ and we are done. Otherwise, let C be a cycle in G. We delete an edge from C as follows.

(i) If C has two consecutive vertices x and y such that $x \notin M$ and $y \notin M$, then delete the edge between them. The set M is still a total dominating set for the resulting graph.

(ii) Suppose the first case does not apply. If C has two consecutive vertices x and y such that $x \in M$ and $y \notin M$, then delete the edge between them. Since the other neighbor of y on C is necessarily in M (or else the first case applies), the set M is still a total dominating set for the resulting graph.

(iii) If neither of the first two cases applies, then all of the vertices of C are in M. Delete any edge of C. The set M is still a total dominating set for the resulting graph.

Repeat this process until all cycles are removed. Call the resulting spanning tree S. Since M is a total dominating set for S, $\gamma_t(S) \leq |M| = \gamma_t(G)$. Since the total domination number of a graph is at most the total domination number of any of its spanning trees, $\gamma_t(G) \leq \gamma_t(S)$. Thus, $\gamma_t(S) = |M|$ and M is a minimum total dominating set of S. ■

Lemma 3: *Let G be a connected graph with $n > 1$. Then*

$$\gamma_t \geq \frac{d+1}{2}.$$

Proof: Let M be a minimum total dominating set of G with k components. Let S be a spanning tree of G, as in Lemma 2, such that M is also a minimum total dominating set of S. Since M induces a forest with k trees, there are $\gamma_t - k$ edges induced by M. By Lemma 1, there are at most $k-1$ edges induced by $S - M$. In traversing a diametric path of S, we can enter and leave each component of M at most once. Thus in a diametric path of S, there are at most $2k$ edges that have an endpoint in M and an endpoint in $S - M$. Noting that $2k \leq \gamma_t$, we get the following upper bound on the diameter $d(S)$ of S:

(5) $$d(S) \leq (\gamma_t - k) + (k-1) + 2k = \gamma_t + 2k - 1 \leq 2\gamma_t - 1.$$

However, since the diameter of a graph is at most the diameter of any of its spanning trees, $d(G) \leq d(S)$ and this completes the proof. ■

Now we are prepared to prove Theorem 3.

Proof of Theorem 3: Let M be a minimum total dominating set of G. Let S be a spanning tree of G, as in Lemma 2, such that M is also a minimum total dominating set of S. Now, by Lemma 3, $d(S) \leq 2\gamma_t - 1$. However, since S is a tree, $2r(S) - 1 \leq d(S)$. From these we get the desired inequality since $r(G) \leq r(S)$. ■

The Sophie heuristic of Graffiti.pc (see [8] for description) conjectured the following interesting (albeit weak) characterization of the case of equality for Theorem 3 (numbers 277 and 278 in [6]). Let D be a subset of the vertex set of a graph G. Then $E_G(D)$ is the set of edges of the subgraph of G induced by D.

Theorem 4: *Let G be a connected graph with $n > 1$ and minimum total dominating set M. Then*

$$\gamma_t = r \textit{ if and only if } |E_G(M)| = \frac{1}{2}r.$$

Proof. Suppose $\gamma_t = r$. Let M be a minimum total dominating set of G with k components. From inequality (5) of Lemma 3,

$$2r - 1 \leq \gamma_t + 2k - 1 \leq 2\gamma_t - 1,$$

which together with our assumption that $\gamma_t = r$ implies

$$r = 2k = \gamma_t. \tag{6}$$

On the one hand, since M is a total dominating set, each component of the subgraph induced by M contains at least two vertices and at least one edge, which implies that $k \leq |E_G(M)|$. On the other hand, no component of the subgraph induced by M contains more than one edge, otherwise the component contains at least three vertices, which contradicts that $2k = \gamma_t$. Thus, $k = |E_G(M)|$, and from (6) it follows that $|E_G(M)| = \frac{1}{2}r$.

Suppose

$$|E_G(M)| = \frac{1}{2}r.$$

Since each vertex of M has degree at least one in the subgraph induced by M, $\sum_{v \in M} deg_M(v) \geq \gamma_t$. Moreover, for the subgraph induced by M we have

$2|E_G(M)| = \sum_{v \in M} deg_M(v)$. Now combining the latter two relations, our assumption for this case, and Theorem 3, we get:

$$r = 2|E_G(M)| = \sum_{v \in M} deg_M(v) \geq \gamma_t \geq r. \qquad \blacksquare$$

The *center* of a graph G, denoted by $C(G)$, is the set of all vertices of minimum eccentricity r. The distance from a vertex v to a set is the smallest distance from v to any of the vertices in the set. The *eccentricity of the center*, denoted by $ecc(C(G))$, is the maximum distance from the center to vertices not in the center. By $ecc(C(S))$ we mean the eccentricity (with respect to S) of the center of the subgraph S of G. When $ecc(C(G)) = r(G)$, the following theorem provides an improvement on Theorem 3.

Theorem 5: *Let G be a connected graph with $n > 1$. Then*

$$\gamma_t \geq 1 + ecc(C(G)).$$

Proof. Let M be a minimum total dominating set of G and let S be the spanning tree formed, as in Lemma 2, such that M is also a minimum total dominating set of S. Since S is a tree, $2r(S) - 1 = d(S)$ or $2r(S) = d(S)$.

Suppose that $2r(S) - 1 = d(S)$. In this case, any diametric path in S is an even path and S has a bi-center (the center is a pair of adjacent vertices). Consequently, $ecc(C(S)) = r(S) - 1$. Thus, using inequality (5) of Lemma 3,

$$1 + ecc(C(S)) = r(S) = \frac{d(S)+1}{2} \leq \frac{(2\gamma_t - 1) + 1}{2} = \gamma_t.$$

On the other hand, suppose $2r(S) = d(S)$. Now, any diametric path in S is an odd path, S has a unique center vertex, and consequently $ecc(C(S)) = r(S)$. Thus, using inequality (5) of Lemma 3,

$$1 + ecc(C(S)) = 1 + r(S) = 1 + \frac{d(S)}{2} \leq \frac{2\gamma_t - 1}{2} = \gamma_t + \frac{1}{2}.$$

This implies that $1 + ecc(C(S)) \leq \gamma_t$ since the left hand side of the above inequality is an integer. So in either case, $1 + ecc(C(S)) \leq \gamma_t$.

We need now only show that the inequality, $ecc(C(G)) \leq ecc(C(S))$, is valid. That is, that the eccentricity of the center of a graph is at most the eccentricity of the center of a spanning tree of the graph. To do this consider the following inequality,

$$ecc(C(G)) \leq r(G) \leq r(S) \leq ecc(C(S)) + 1.$$

Suppose that $ecc(C(G)) = ecc(C(S)) + 1$. This implies that all of the above are equal. In particular, since $ecc(C(S)) + 1 = r(S)$, S is a bi-centric tree. Let $\{x, y\}$ be the bi-center of S. Moreover, let $d_G(p, q)$ denote the distance from p

to q in G. Since for any vertex w in G,

$$d_G(x,w) \leq d_S(x,w) \leq r(S) = r(G),$$

we conclude that x is also center of G. Similarly, y is also a center of G.

Let z be a vertex at eccentric distance from $C(G)$ in G. Note that for any v in $C(G)$,

$$d_G(v,z) \geq ecc(C(G)) = ecc(C(S)) + 1.$$

Now because x is a center of G,

$$r(G) = r(S) \geq d_S(x,z) \geq d_G(x,z) \geq ecc(C(G)) = r(G),$$

and we conclude that $d_S(x,z) = r(S)$. Similarly, $d_S(y,z) = r(S)$. However, this situation is impossible because only one of these equations can be true for a bi-centric tree. Hence, it must be the case that $ecc(C(G)) < ecc(C(S)) + 1$, which proves our claim. ■

The *girth* $g = g(G)$ is the minimum order of an induced cycle in a graph G containing a cycle. It is easy to show that the girth of a graph can be used to provide a lower bound for the total domination number (Graffiti.pc's Conjecture 249 in [6]). Occasionally this lower bound may be slightly better than that given by the radius.

Proposition 1: *Let G be a graph containing a cycle. Then*

$$\gamma_t \geq \frac{g}{2}.$$

Proof. We can dismiss the case $g \leq 4$ handily, since $\gamma_t \geq 2$. Thus suppose $g > 4$. Let M be a minimum total dominating set. Let C be a cycle of minimum order and let K be the intersection of M and C. We can assume that $|K| < \frac{g}{2}$, since otherwise the inequality is trivial. Since each vertex of K totally dominates two vertices of C, at most $2|K|$ vertices of C are totally dominated by vertices from K. Each vertex of C not totally dominated by a vertex in K must be totally dominated by a distinct vertex of M outside of C, since two or more of these vertices could not have been totally dominated by the same vertex of $M - K$ or a shorter cycle is present. This yields:

$$\gamma_t \geq |K| + g - 2|K| = g - |K| > \frac{g}{2}.$$ ■

The characterization of the case of equality for Proposition 1 can easily be derived from its proof. For $g \leq 4$, this characterization is similar to the characterization of graphs where $\gamma_t = 2$: $\gamma_t = \dfrac{g}{2}$ if and only if there exists and edge $\{x, y\}$ such that 1) $N(x)$ and $N(y)$ are both independent sets; 2) $N(x)$ and $N(y)$ are disjoint and their union is G; and 3) at least one vertex of $N(x)$ is adjacent to at least one vertex of $N(y)$.

On the other hand, for $g \geq 5$, $\gamma_t = \dfrac{g}{2}$ if and only if $g \equiv 0 \bmod 4$ and there exists an induced cycle of order g whose edges can be labeled clockwise $1, 2, \ldots, g$ such that all non-cycle vertices have degree 1 and are incident to cycle edges that have labels from the same congruence class mod 4.

We let $L = L(G)$ denote the maximum number of leaves (vertices of degree 1) over all spanning trees of G and $l = l(G)$ denote the minimum number of leaves over all spanning trees of G. Graffiti.pc's Conjecture 297 in [6] asserts that $\gamma_t + \dfrac{1}{2}\gamma_c \leq n$. A simple known fact is that the connected domination number and L are related by $\gamma_c = n - L$, and thus 297 is equivalent to the statement that the total domination number is bounded above by the average of n and L. The authors found two independent proofs of this conjecture. Below, we observe that this conjecture is also a corollary to a result of M. Chellali and T. Haynes found in [1], which we state next along with another of their results found in [2]. In a tree, a vertex adjacent to a leaf of the tree is called a *support vertex.*

Theorem 6 *(M. Chellali and T. Haynes [1], [2]): Let T be a tree with $n > 2$ vertices, l leaves, and s support vertices. Then*

$$\frac{n+2-l}{2} \leq \gamma_t \leq \frac{n+s}{2}.$$

Corollary 1: *Let G be a connected graph with $n > 1$. Then*

$$\gamma_t \leq \frac{n+l}{2}.$$

Proof. The case $n = 2$ is obvious. Otherwise, let T be a spanning tree of G with l leaves and s support vertices. Then

$$\gamma_t(G) \leq \gamma_t(T) \leq \frac{n+s}{2} \leq \frac{n+l}{2}. \qquad \blacksquare$$

A subset of the edges of a graph G such that no two edges are incident is a *matching* in G. A *maximal matching* is a matching that is not contained in a larger matching; let $\mu^* = \mu^*(G)$ denote the cardinality of a minimum maximal matching. The number of edges in a maximum matching is the *matching number*, which is denoted by $\mu = \mu(G)$. A graph is *claw-free* if it contains no induced

$K_{1,3}$ (the complete bipartite graph with partitions of size one and three). It is known that whenever a graph is claw-free and of minimum degree at least three, the total domination number is bounded above by the matching number (see [17]).

A collection of vertex disjoint paths of a graph G that partition the vertices of G is a *path covering* of G. The cardinality of a minimum path covering is denoted by ρ=$\rho(G)$. Note that $\rho = 1$ if and only if the graph has a Hamiltonian path. Graffiti.pc conjectured an upper bound on $\gamma_t(G)$ involving the matching and path covering numbers of a G (number 288 in [6]), which we prove in the next theorem. Let C_m, K_m and P_m be the cycle, complete graph and path on m vertices, respectively. Moreover, note that the bound is sharp for every value of ρ, as demonstrated by taking C_m with $m \geq 1$, using the assumption that $C_1 = K_1$ and $C_2 = P_2$, and identifying each vertex of the cycle with the center of a copy of P_7. Let the constructed graph be called G_m, then $\gamma_t(G_m) = 4m$, $\mu(G_m) = 3m$, and $\rho(G_m) = m$.

Theorem 7: *Let G be a connected graph with* $n > 1$. *Then*

$$\gamma_t \leq \mu + \rho.$$

Proof. Let $\mathcal{P} = \{P_1, P_2, ..., P_\rho\}$ be a minimum path covering of G with P_i having n_i vertices. Starting from one end, let M_i be the matching consisting of the edges in odd position along P_i, so that $|M_i| = \left\lfloor \frac{n(i)}{2} \right\rfloor$. For each i such that $n_i > 1$, we construct a total dominating set D_i for P_i such that $|D_i| \leq |M_i| + 1$. If $n_i > 1$ for $1 \leq i \leq \rho$, then this completes the proof, since it yields a total dominating set for G with size at most $\mu + \rho$.

In general, to form D_i we take the edges of M_i in pairs from the beginning, putting into D_i the two central vertices in this set of four vertices along P_i. If $n_i \equiv 0 \mod 4$, this works very simply, with $|D_i| = |M_i| = \frac{n(i)}{2}$. In other congruence classes, we must be careful to dominate the vertices at the end, after the last group of four vertices. If $n_i \equiv 1 \mod 4$, then it suffices to add the next-to-last vertex on P_i to D_i, yielding $|D_i| = |M_i| + 1$. This works because the vertex before it is also in D_i. If $n_i \equiv 2 \mod 4$, then we instead add the last two vertices. They comprise the last edge of M_i, so again $|D_i| = |M_i| + 1$. If $n_i \equiv 2 \mod 4$, those two vertices we just added also take care of the last vertex.

As remarked earlier, the proof is now complete unless some paths in the partition are isolated vertices; we index the paths so that these are $P_{k+1}, ..., P_\rho$. Let $D = \bigcup_{i=1}^{k} D_i$ and $M = \bigcup_{i=1}^{k} M_i$, we have $|D| \leq |M| + k$. Consider P_j with $j > k$; let v be the one vertex of P_j. Since G is connected and we have a minimum path covering, v has a neighbor x on some path P_i with $i < k$. Since D_i is a total dominating set for P_i, we can add x to D_i (if it is not already there) to dominate v. After doing this for each j with $k < j < \rho$, we have constructed a total dominating set of G with size at most $\mu + \rho$. ■

Graffiti.pc's Conjecture 247 asserts that $\gamma_t \geq 2\rho$ when G is a regular graph. The desired inequality does not hold for all graphs. In particular, consider $K_{m,n}$. If $|m - n| > 1$, then at least two paths are needed to cover the vertices, but the total domination number is two.

On the other hand, the inequality is trivial for k-regular graphs with $k \in \{1, 2\}$. Each component takes one path to cover and contributes at least two vertices to a total dominating set. Equality holds when $k = 1$ or when $k = 2$ and all components are either C_3 or C_4.

We prove the inequality for cubic graphs. We will use a given total dominating set S to construct a path partition of $V(G)$ such that each vertex of S is associated with one path, and on average at least two vertices of S are associated with each path. Note that the bound is sharp infinitely often, in particular whenever each component is isomorphic to $K_{3,3}$ or $K_3 \square K_2$ (the cartesian product of K_3 and K_2); moreover, we conjecture that equality holds only in this case.

Theorem 8: *Let G be a connected 3-regular graph. Then*

$$\gamma_t \geq 2\rho.$$

Proof: Let S be a total dominating set (*td-set* for convenience), and let $H = G - S$. Since each vertex outside S has a neighbor in S, we have $\Delta(H) \leq 2$. Hence each component of H is a path or a cycle.

We construct a path partition, in two phases. In the first phase, we construct pairwise disjoint paths that together include all of $V(H)$ and some vertices of S. The paths have vertices of S at both ends, and no two consecutive vertices along one of these paths belong to S. Each step of Phase 1 absorbs one component of H, producing a family of paths with these properties. Let $\mathcal{P}$ be the current family. Let S' be the set of vertices of S that appear on the paths in $\mathcal{P}$.

Case 1 of Phase 1: a component of H that is a path P. Let x and y be the endpoints of P (possibly $x = y$). Each endpoint of P has at least two neighbors in S (three if P is an isolated vertex of H). Choose $u \in N(x) \cap S$ and $v \in N(y) \cap S$ with $u \neq v$. This can be done easily if $x = y$. Note that if u is an endpoint of a path in P, then u cannot be adjacent to both x and y, since S is a td-set; similarly for v. Indeed, the only case where both neighbors of x in S can also be neighbors of y is when those vertices are not yet in S'; we simply let one be u and the other be v.

Vertices u and v may or may not lie in S'. Since S is a td-set, neither u nor v can be an internal vertex of a path in $\mathcal{P}$, since such vertices have two neighbors already outside S. Thus each of u and v is an endpoint of a path in $\mathcal{P}$ or is as yet unused in S'. If u and v are endpoints of the same path P' in $\mathcal{P}$, then x is not adjacent to v, since S is a td-set. Hence x has a neighbor outside $\{u, v\}$ in S, and we use that vertex instead of u.

Hence u and v are not endpoints of the same path in $\mathcal{P}$. Combine P with the edges ux and yv and the paths in $\mathcal{P}$ (if they exist) that are already associated with u and/or v to form a single path that has the desired properties (replacing the paths used that were in $\mathcal{P}$).

Case 2 of Phase 1: a component of H that is a cycle C. Each vertex x on C has exactly one neighbor in S, since S is a dominating set and x has two neighbors already on C. No vertex of S has three neighbors on C, since S is a td-set. Since C has at least three vertices, we can therefore find two consecutive vertices on C (call them x and y) whose neighbors in S (call them u and v, respectively) are distinct.

As in Case 1, neither u nor v is an internal vertex of a path in $\mathcal{P}$. If we can choose x and y above so that u and v are distinct and are not the endpoints of a single path in $\mathcal{P}$, then we can absorb the path $C - xy$ as described in Case 1.

On the other hand, if these neighbors u and v are endpoints of the same path P' in P, then let x' be the neighbor of y on C other than x, and let u' be the neighbor of x' in S. Since v already has two neighbors not in S (both y and the neighbor of v on P'), we cannot have $u' = v$, since S is a td-set. Similarly $u' \neq u$. Now u' is not an endpoint of P', and we can absorb the path $C - yx$ as described in Case 1.

Phase 2: All of H has been absorbed. Recall that S' denotes the subset of S that has been used on the paths in $\mathcal{P}$. These paths each have at least two vertices of S' and cover all of $V(G)$ except $S - S'$. Let M be a maximum matching in the subgraph of G induced by $S - S'$. Each edge of M is a path with two vertices of S; we add this path to our family $\mathcal{P}$. It remains only to absorb the vertices of $S - S' - V(M)$.

Let $T = S - S' - V(M)$. By the choice of M, the set T is independent in G. Since S is a td-set, each vertex w in T has at least one neighbor in S. Choose one such neighbor arbitrarily; it lies on a path in $\mathcal{P}$. We absorb these vertices of T into our path partition.

Let P be a path in $\mathcal{P}$. The chosen edges joining P to vertices of T form a caterpillar with P. Each internal vertex of P in S' can have one such neighbor; the endpoints of P can have two. If an endpoint acquires two new neighbors, we use one to extend P, and the other becomes a path of length 0. This does not cause a problem, because we have increased the number of components of the path partition by 1 while absorbing two additional vertices of T. If the endpoint has one new neighbor, we just extend the path.

It remains to consider internal vertices of the original path that are selected from T; each such vertex is selected at most once. If j internal vertices of P are selected, then we have $2j$ "extra" vertices of S associated with P (in addition to the endpoints and the vertices possibly appended to the endpoints), so we can afford to cut the path j times (just before each internal vertex receiving a new neighbor), creating j additional paths in the path partition but having the total number of vertices of S associated with these paths be at least twice the number of paths. ■

The following theorem was inspired by Graffiti.pc's Conjecture 246, that the total domination number in trees is at least one less the matching number. It is not too difficult to see that this conjecture is false, however, the following theorem shows that the total domination number and the minimum size of a maximal matching (denoted μ*) are related.

Theorem 9 : *Let T be a tree with $n > 1$. Then*

$$\gamma_t \geq \mu* + 1.$$

Proof. Let T be a tree, and let D be a minimum total dominating set of T with γ_t vertices and k components. We will build a maximal matching M with at most $\gamma_t - 1$ edges. To start, take a maximal matching from the forest induced by D and call this set M_1. Next, take a maximal matching from the forest induced by $T - D$ and call this set M_2. Finally, for each vertex of D not in M_1, it may be possible to match that vertex to a currently un-matched vertex of $T - D$. Let M_3 be the set of all such possible edges and set $M = M_1 \cup M_2 \cup M_3$. By this construction, M is maximal.

Now, from Lemma 1 we see,

$$|M_2| \leq k - 1.$$

In addition, we can bound the number of edges in M_3 with the inequality,

$$|M_3| \leq \gamma_t - 2|M_1|.$$

Therefore,

$$\begin{aligned}\mu* \leq |M| = |M_1| + |M_2| + |M_3| &\leq |M_1| + k - 1 + \gamma_t - 2|M_1| \\ &= \gamma_t - 1 - (|M_1| - k).\end{aligned}$$

From this we get the desired inequality $\mu* \leq \gamma_t - 1$, since $|M_1| \geq k$. ■

An assignment of k colors to the vertices of a graph G such that adjacent vertices are assigned different colors is a k-coloring of G. The minimum k for which a graph has a k-coloring is called the *chromatic number* and is denoted by $\chi = \chi(G)$. Complete graphs and trees demonstrate that the total domination number is not bounded below or above by the chromatic number. Graffiti.pc's Conjecture 228 in [6] states that when G is triangle-free, the total domination number is indeed bounded below by the chromatic number.

Proposition 2: *Let G be a triangle-free graph with* $n > 1$. *Then*
$$\gamma_t \geq \chi\,.$$

Proof. Let S be a smallest total dominating set. It suffices to cover V with $|S|$ independent sets. Since G is triangle-free, the neighborhood of each vertex of S is an independent set. Since S is a total dominating set, the union of these neighborhoods is V. ■

The local independence at a vertex v of a graph G is the independence number of the subgraph induced by the neighbors of v. We use $\lambda = \lambda(G)$ as the maximum of local independence over all vertices of G. Note that $\lambda(G) = 2$ if and only if G is claw-free and is not a complete graph. The order of a largest complete subgraph is known as the clique number and denoted by $\omega = \omega(G)$. The clique number of a graph does not bound the total domination number above as seen by the following construction. Take K_m for $m \geq 3$ and add a pendant edge at each, then add a pendant edge at one of the resulting vertices of degree 1. The resulting graph has total domination number $m + 1$ while the clique number is m. This family of graphs also demonstrates that the bound in the next theorem (Graffiti.pc number 301), involving the clique number of the graph and the maximum of local independence of the complement graph, is sharp.

Theorem 10: *Let G be a connected graph with* $n > 1$. *Let*
$A = \{v :$ *local independence of* v *in* G^c *is maximum*$\}$. *Then*

$$\gamma_t \leq \omega + |A|.$$

Proof. We can assume that G is not complete, since the relation holds otherwise. Let v be a vertex of maximum local independence in G^c.

Observation. A vertex v of maximum local independence $\lambda(G^c)$ in G^c has the property that in G there exists a clique of order $\lambda(G^c)$ whose vertices are not adjacent to v.

By the above observation, there exists a clique K of order $\lambda(G^c)$ in G such that no vertex of K is adjacent to v in G. Clearly $V(K)$ is a total dominating set for the subgraph induced by $N(K)$. By assumption, v is in $V(G) - N(K)$. Since each vertex in $V(G) - N(K)$ has local independence at least $\lambda(G^c)$ in G^c, each is a vertex of maximum local independence in G^c, that is,

$$|V(G) - N(K)| \leq |A|.$$

In the case that $V(G) - N(K)$ is a total dominating set for the subgraph induced by $V(G) - N(K)$, it follows that $V(K) \cup [V(G) - N(K)]$ is a total dominating set for G, and

$$\gamma_t \leq |V(K)| + |V(G) - N(K)| \leq \omega + |A|.$$

On the other hand, in case $V(G) - N(K)$ is not a total dominating set for the subgraph induced by $V(G) - N(K)$, there must exist an isolated vertex x in the subgraph induced by $V(G) - N(K)$. Since G is assumed to be connected, x must be adjacent to some vertex n in $N(K) - V(K)$. Now let $x_1, x_2, ..., x_k$ be the isolated vertices in the subgraph induced by $V(G) - N(K)$. For each vertex x_j, let n_j be a neighbor of x_j in $N(K)$ with respect to G. Then $[V(G) - N(K)] - \{x_1, x_2, ...x_k\} \cup \{n_1, n_2, ..., n_k\}$ is a total dominating set for the subgraph induced by $V(G) - N(K)$. Finally, $V(K) \cup [[V(G) - N(K)] - \{x_1, x_2, ...x_k\} \cup \{n_1, n_2, ..., n_k\}]$ is a total dominating set for G of order at most $\omega + |A|$. ■

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