Graffiti.pc on the total domination number of a tree

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Abstract

The total domination number of a simple, undirected graph G is the minimum cardinality of a subset D of the vertices of G such that each vertex of G is adjacent to some vertex in D. In 2007 Graffiti.pc, a program that makes graph theoretical conjectures, was used to generate conjectures on the total domination number of connected graphs. More recently, the program was used to generate conjectures on the total domination number of trees. In this paper, we discuss and resolve several of these conjectures for trees, which are often improvements over known results for all connected graphs.
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1 Introduction and Definitions

A subset D of the vertices of a simple, undirected graph $G = (V, E)$ is a *total dominating set* if each vertex in V is adjacent to some vertex in D. The *total domination number* of a graph G, denoted $_t(G)$, is the minimum cardinality of a total dominating set of G. Total domination in graphs was introduced in 1980 by Cockayne, Dawes, and Hedetniemi [3]. Since then a number of papers on total domination in graphs have been published. Comprehensive surveys appeared in 1998 (see [10] and [11]), and more recently in [12].

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The total domination problem is known to be NP-complete and attributed to Pfaff, Laskar and S. Hedetniemi [14]. Its complexity is one motivation for discovering bounds for the total domination number of a graph. For trees, on the other hand, there exists a linear algorithm [13], thus bounds on the total domination number do not seem to have the same immediate motivation. However, observe that since there is a spanning tree of a connected graph that has the same total domination number as the graph, some bounds on $_t$ for trees are useful in proofs for bounds on $_t$ for connected graphs (as was the case in Theorems 3 and 5 in [4].)

Graffiti.pc is a program that makes graph theoretical conjectures. In 2007, DeLaViña used Graffiti.pc to generate conjectures involving the total domination number of a connected graph [4], and more recently to generate conjectures involving the total domination number of trees (the topic of this paper.) These conjectures take the form of upper or lower bounds for the total domination number. In addition to their potential usefulness in proving bounds on total domination for graphs in general described above, we found the conjectures for trees aesthetically appealing, as they sometimes provide clever or surprising improvements over known bounds for all connected graphs. Moreover, their proofs (or counter-examples) can often be challenging or amusing. A numbered, annotated listing of Graffiti.pc's total domination conjectures and their current status can be found in [6]. Graffiti.pc employs two main strategies (called Dalmatian and Sophie) for generating conjectures. The principle behind the Dalmatian heuristic (used for conjectures discussed in this paper) is due to S. Fajtlowicz and its implementation within Graffiti.pc is discussed in [5].

Let G be a graph with vertex set $V = V(G)$. The number of vertices of G we denote by $n(G)$. The degree sequence of a graph provides many graph invariants, including its *maximum degree* and *minimum degree*, which we denote by $\Delta(G)$ and $\delta(G)$, respectively. The number of distinct values that occur in the degree sequence is called the *number of distinct degrees of* G and is denoted by $dd(G)$. In a graph G, a vertex of degree zero is called an *isolated vertex.* A vertex of degree one in a tree is called a *leaf*, and a vertex that is adjacent to a leaf is called a *support vertex.*

For a subset $A \subset V$, let $N(A)$ denote the *neighborhood of* A, that is, the set of vertices adjacent to vertices in A. Let $G[A]$ denote the *subgraph of* G *induced by* A.

Finally, let $\ = \ (G)$ denote the independence number of G, that is the maximum cardinality of a subset of pairwise non-adjacent vertices.

The following proposition is a summary of some easily deduced facts that will be used in this paper.

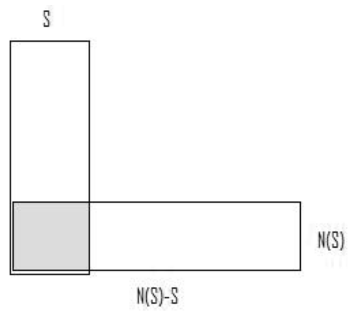

Figure 1: Neighborhoods of support vertices

Proposition 1. *Let G be a graph with no isolated vertices. Then*

$$_t(G) \geq \frac{n(G)}{\Delta(G)} \tag{1}$$

$$_t(G) \leq 2 \quad (G) \tag{2}$$

$$_t(G) \leq n(G) - \Delta(G) + 1 \tag{3}$$

2 Results

Graffiti.pc's conjecture #331 in [6] states that the total domination number of a tree is at most twice its independence number minus the number of isolated vertices induced by the neighborhood of its support vertices, which suggests an improvement for trees on the second upper bound given in Proposition 1. Let us observe the following about the neighborhood of support vertices of trees (Figure 1 makes it obvious).

Observation 1. *Let T be a tree and S its support vertices. Then the number of isolated vertices induced by $N(S)$ is less than equal to the number of isolated vertices induced by $N(S) - S$.*

This observation together with Theorem 2 settles and improves on Conjecture #331.

Theorem 2. *Let T be a non-trivial tree and S its support vertices. Then $_t \leq 2 \quad - |L^*|$, where L^* is the set of vertices in $N'(S) = N(S) - S$ with degree 0 with respect to $T[N'(S)]$.*

Proof. The theorem is obvious if T is a star, so let's assume otherwise. Let $N'(S) = N(S) - S$. So $N'(S)$ and S are disjoint. Let L denote the set of leaves of T, and L^+ denote the vertices of $N'(S)$ that are not leaves

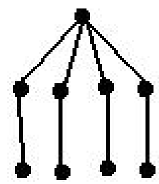

Figure 2: spider(4) has $\ \ = L^* = \ _t = 5$

but have degree 0 with respect to $T[N'(S)]$. So clearly $L^*= L \cup L^+$. Put $M = N'(S) - (L \cup L^+)$, and let M^* be a maximum independent set of $T[M]$. Let $M' = M - M^*$. Because $T[M] = T[M^* \cup M']$ is a forest, we have $|M'| \le |M^*|$. Next, consider the set $L \cup L^+ \cup M^*$. Although it is easy to see this set is independent in T, it may not be a maximal independent set. Hence, let K be a set of vertices such that $I^*= L \cup L^+ \cup M^* \cup K$ is a maximal independent set of vertices in T. Note that the component sets L, L^+, M^*, and K of I^* are pairwise disjoint. Moreover, by construction, every vertex of K must be adjacent to some vertex in M'.

Put $D = S \cup M' \cup L^+ \cup M^* \cup K$. We observe the component sets S, M', L^+, M^*, and K of D are pairwise disjoint. Let $K' = V(G) - (K \cup S \cup N'(S))$. So $V(G) = S \cup N'(S) \cup K \cup K' = S \cup L^+ \cup L \cup M^* \cup M' \cup K \cup K'$, and D consists of all the vertices in T other than those in $L \cup K'$. D is a total dominating set for T. Every vertex in K' must be adjacent to some vertex in $L^+ \cup M^* \cup K$, by the maximality of the independent set I^*. If there is a vertex in S which is adjacent only to vertices in L, then T is a star, contrary to our assumption. So every vertex in S must be adjacent to a vertex in $(N'(S) - L) \subseteq D$. M', L^+, and M^* are all subsets of $N'(S) - L$, so every vertex in these sets is adjacent to a vertex in $S \subseteq D$. Finally, every vertex in K is adjacent to a vertex in $M' \subseteq D$.

Now,

$\ _t \le |D|$
$= |S \cup M' \cup L^+ \cup M^* \cup K|$
$= |S| + |M'| + |L^+| + |M^*| + |K|$
$\le |L| + |M^*| + |L^+| + |M^*| + |K| + |K|$
$= 2|L| + 2|L^+| + 2|M^*| + 2|K| - |L| - |L^+|$
$= 2(|L| + |L^+| + |M^*| + |K|) - (|L| + |L^+|)$
$= 2|L \cup L^+ \cup M^* \cup K| - |L \cup L^+|$
$= 2\ \ ^* - |L^*|$
$\le 2\ \ - |L^*|$ □

The bound in Theorem 2 is sharp for *spider(k)*, the tree on $2k+1$ vertices constructed by identifying an endpoint of each of k paths on 3 vertices (see Figure 2). This is because the independence and total domination numbers are equal to $k+1$, and the number of isolated vertices induced by support vertices is also $k+1$.

The *eccentricity of a vertex* v of a graph, denoted $\hat{r}(v)$ is the maximum distance from v to another vertex of the graph. The *radius* of a graph G is the minimum eccentricity among vertices of G, and the *diameter* of a graph is the maximum eccentricity among vertices of G, denoted $rad(G)$ and $diam(G)$, respectively. The *boundary* (or *periphery*) is the set of vertices of maximum eccentricity, and the *center* is the set of vertices of minimum eccentricity. The *eccentricity of a set* X is the minimum eccentricity of vertices in X, that is the distance of a vertex furthest away from all vertices of X. In [4] we proved that the total domination number of a connected graph G is at most one plus the eccentricity of the center of G, which we denote by $\hat{r}(G)$; we restate this fact next as Theorem 3.

Theorem 3. *[4] Let G be connected graph on $n \geq 2$ vertices. Then* $_t(G) \geq \hat{r}(G) + 1$.

Graffiti.pc's #357 proposes that for trees this bound can be improved.

Theorem 4. *Let T be a non-trivial tree. Then* $_t(T) \geq \hat{r}(T) + |N(B)| - 1$, *where* $B = \{v \mid \hat{r}(v) = diam(T)\}$.

Proof. For ease of notation, let $N_b(T)$ represent the number of neighbors of boundary vertices of T, and thus the number of support vertices which have neighbors in the boundary set. Proceeding by induction on N_b, notice that if $N_b = 2$, the result follows from Theorem 3, settling our base case.

Assume the theorem is true for all trees with $N_b = k \geq 2$ and let T be a tree with $N_b = k+1$. Let P be a diametral path of T with end vertices x and y, and let v be a leaf boundary vertex whose support vertex is not on P (such a vertex exists since $N_b \geq 3$).

Without loss of generality assume $d(v,x) \geq d(v,y)$. Let u be the closest vertex to v on P. Now, since v is also on a diametral path and all diametral paths must contain the center (or bi-center) of a tree, we can deduce each of the following is true, where a nontrivial branch point is a vertex with at least three neighbors of degree two or more.

(i) $d(v,u) = d(u,y)$

(ii) $d(x,y) = diam = d(x,v)$

(iii) There is a nontrivial branch point on the path from v to u. Let w be the closest of these to v. Note that $d(v,w) \geq 2$.

Next, let e be the edge adjacent to w on the path from w to v and let C be the component of $T-e$ containing v. Now call T' be the subtree $T-C$. Let D be a minimum total dominating set of T containing no leaves, and D' the vertices of D in T'. We make the following observations about T'.

(iv) $\hat{r}(T') = \hat{r}(T)$

(v) $N_b(T') = N_b(T) - 1$

Suppose $d(v, w) \geq 3$. Since there are at least two vertices in D that are not in D' and we can extend D' to a total dominating set of T' by adding at most one vertex, ${}_t(T') \leq {}_t(T) - 1$. On the other hand, suppose $d(v, w) = 2$. Let z be a non-leaf neighbor of w, not on the path from v to x. Thus, z must be a support vertex and consequently in D'. Therefore D' is a total dominating set of T' , and so we still have ${}_t(T') \leq {}_t(T) - 1$. Finally, taken together and applying our inductive hypothesis,

$$(T) \geq (T') + 1 \geq \hat{r}(T') + N_b(T') - 1 + 1 = \hat{r}(T) + N_b(T) - 1.$$

□

In [2], M. Chellali and T. Haynes proved that the total domination number of a tree is bounded below by half of two more than the number of non-leaf vertices. The number of non-leaf vertices in a tree is precisely the number of cut vertices of the tree, and thus a corollary to their result is that the total domination number of a tree is at least one plus half the number of cut vertices. Now Graffiti.pc's #355 is a corollary to the latter and Theorem 6.

Theorem 5. *(M. Chellali and T. Haynes [2]) Let T be a non-trivial tree. Then ${}_t(T) \geq \frac{n(T)-L+2}{2}$.*

Theorem 6. *Let T be non-trivial tree such that $\hat{r}(T) = 2$. Then ${}_t(T) \geq x(T)$, where $x(T)$ is the number of cut vertices of T.*

Proof. Let D be a minimum total dominating set containing no leaves (which must exist since T is not a star.) Each support vertex is in D and each center has a support neighbor since $\hat{r}(T) = 2$. Thus each center is in D to dominate its support neighbors. Since every non-leaf is a center or a support vertex, the result follows. □

Corollary 7. *Let T be a non-trivial tree such that $\hat{r}(T) \geq 2$. Then ${}_t(T) \geq \frac{x(T)}{\hat{r}(T)-1}$, where $x(T)$ is the number of cut vertices of T*

Lemma 8. *Let T be a non-trivial tree with the property that the vertices of T can be partitioned into two sets A and B such that the vertices in A are of degree two and B is an independent set. Then ${}_T(T) \geq |B|$.*

Proof. Let D be a minimum total dominating set for T. Enumerate the vertices in B as $v_1, ..., v_k$. Now for each v_i, if v_i is in D, let $d_i = v_i$. Otherwise, let d_i be some neighbor of v_i in D. Now suppose $d_i = d_j$ for

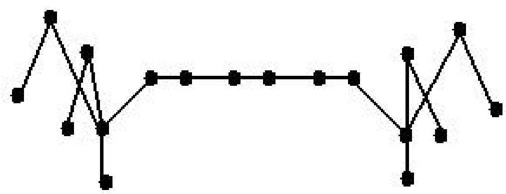

Figure 3: G(2) has $_t = 8$, $c_2 = 6$, and $p_2 = 6$

distinct values of i and j. If $v_i = d_i$, or $v_j = d_j$, then v_i and v_j must be adjacent, a contradiction. So we can assume $d_i = d_j$ is a vertex in A, i.e. it has degree 2. But then one of either v_i or v_j must be in D, a contradiction of our choice of $d_i = d_j$. Thus $|D| \geq |\{d_1, ..., d_k\}| = |\{v_1, ..., v_k\}|$. $\square$

P. Feit discovered a 34-vertex counter-example to Graffiti.pc's Conjecture #240 and consequently Conjecture #346 [6]. Indeed, his counter-example can be extended to make the left and right sides of these inequalities arbitrarily far apart. Conjecture #240 says, for example, that for a tree the total domination number is at least the number of components induced by non-degree two neighbors of degree two vertices. In Theorem 9 we prove a weaker version of Conjecture #346. Theorem 9 is sharp for many trees. For instance, let $G(k)$ be the graph constructed by taking a path on 10 vertices (enumerated left to right as $1, 2, ...10$) and identifying an endpoint of each of k paths on 3 vertices with vertex 2 on the 10-path and similarly identifying an endpoint of each of k paths on 3 vertices with vertex 9 on the 10-path. Observe that for $k \geq 1$: $_t(G(k)) = 2k + 4$; the number of components of the subgraph induced by the complement of the degree two vertices, denoted by c_2, is $2k + 2$; and the order of largest component induced by degree two vertices, denoted by p_2, is 6. See Figure 2 for $G(2)$.

Theorem 9. *Let T be a non-trivial tree and S the set of vertices of degree two. Let c_2 be the number of components of the subgraph induced by $V(T) - S$, and p_2 the order of a largest path in the subgraph induced by S. Then*

$$_t(T) \geq c_2 + \frac{p_2}{2} - 1.$$

Proof. We assume that T has at least 2 degree two vertices otherwise the relation follows trivially. Let us derive a tree T' from T as follows. For each component of $T[V(T) - S]$, contract the vertices of the component to a single vertex v_i such that an edge in T incident to a vertex of the component and to a vertex u in no component of $T[V(T) - S]$ is still incident to u but now also to v_i. Let us call the set of contracted component vertices

$B = \{v_1, v_2, ..., v_{c_2}\}$. Observe that components of $T[S]$ are paths and let p_2 be the order of a longest such path P. Next, let P' be the path P without the endpoints of P and add to B the vertices of a largest independent set of $T[P']$. Let $A = V(T') - B$. Observe that the vertices in A are of degree two in T', and that B is an independent set in T'. By construction and Lemma 8, $c_2 + \frac{p_2-2}{2} \leq |B| \leq {}_t(T')$. □

Lemma 10. *Let T be a non-trivial tree. Let D be a minimum total dominating set of T. Then for any vertex $v \in V(T) - D$,*

$$deg(v) \leq \frac{{}_t(T)}{2}.$$

Proof. Let k be the number of components induced by D. For $v \in V(T) - D$, let A be the set of components of D that have a vertex adjacent to v and $B = N(v) \cap (V(T) - D)$ (i.e. the neighbors not in D). Since every vertex in $V(T) - D$ must be adjacent to a vertex in D, in particular a vertex in B must have a neighbor in D. But since T is a tree, no vertex in B is adjacent to a vertex of a component in A nor to two vertices of a component in D. Thus $|B| \leq k - |A|$ and the result follows since $k \leq \frac{{}_t(T)}{2}$ and $deg(v) = |A| + |B|$. □

Theorem 11. *Let T be a non-trivial tree. Then*

$${}_t(T) \geq \frac{2}{3} dd(T).$$

Proof. Let D be a minimum total dominating set of T. Since the vertices in D clearly contribute at most ${}_t(T)$ distinct degrees and by Lemma 10 the vertices in $V(T) - D$ contribute at most $\frac{(T)}{2}$ distinct degrees, $dd(T) \leq \frac{3}{2}\ {}_t(T)$. □

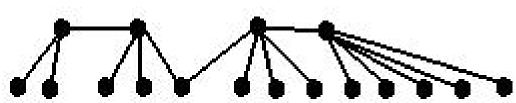

Figure 4: A graph in T_4.

Next we define three classes of trees that comprise all trees for which the bound in Theorem 11 is sharp. By a *binary star* we mean the tree obtained by joining the centers of two stars. Let $B(m_1, m_2)$ be the binary star with support vertices of degrees m_1 and m_2. We define T_2 as follows.

$$T_2 = \{B(m_1, m_2) : m_1 \neq m_2\}$$

i.e. T_2 is the collection of binary stars whose support vertices are of distinct degrees.

Next, let T_4 be the class of graphs constructed by taking the union of $B(m_1, m_2)$ and $B(m_3, m_4)$ with $m_i \geq 3$ and $m_i \neq m_j$ for $1 \leq i, j \leq 4$, and either identifying one leaf in $B(m_1, m_2)$ with one leaf in $B(m_3, m_4)$ (see Figure 4 an example of such a graph), or joining one leaf in $B(m_1, m_2)$ to one in $B(m_3, m_4)$. Observe that for a graph T in T_4, ${}_t(T) = 4$, $dd(T) = 6$ and thus the bound in Theorem 11 is sharp for T.

Lastly, let T_6 be the class of graphs constructed by taking the union of $B(m_1, m_2)$, $B(m_3, m_4)$ and $B(m_5, m_6)$ with each $m_i \geq 4$ and $m_i \neq m_j$ for $1 \leq i, j \leq 6$. Then either identify one leaf in $B(m_1, m_2)$ and $B(m_3, m_4)$, and join this identified vertex to a leaf of $B(m_5, m_6)$, or join one leaf in $B(m_1, m_2)$ to a leaf in each of $B(m_3, m_4)$ and $B(m_5, m_6)$. For a graph T in T_6 of the first type, ${}_t(T) = 6$, $dd(T) = 9$ and thus the bound in Theorem 11 is sharp for T.

Lemma 12. *Let T be a non-trivial tree and let D be a minimum total dominating set of T. If there is $v \in V(T) - D$ such that $deg(v) = \frac{{}_t(T)}{2}$, then $dd(T) \leq {}_t(T) + 3$.*

Proof. Let v be in $V(T) - D$ of degree ${}_t(T)/2$. Let A be the set of components of D that are incident to v and let B be the set of vertices of $V(T) - D$ that are adjacent to v. Since each vertex in $V(T) - D$ must be incident with one component of D and no two vertices in $V(T) - D$ are incident to a common component of D, each vertex in B must be incident with exactly one component of D that is not in A. Moreover, since each vertex of $V(T) - D$ is adjacent to exactly one vertex of a component in D, $|A| + |B| = {}_t(T)/2$ and no other vertex of $V(T) - D$ can be incident with two components of D. Thus, the vertices of $V(T) - D$ contribute at most three distinct degrees, which together with the ${}_t(T)$ possible distinct degrees of the vertices D yields $dd(T) \leq {}_t(T) + 3$. $\square$

Theorem 13. *Let T be a non-trivial tree. Then ${}_t(T) = \frac{2}{3}dd(T)$ if and only if $T \in T_2 \cup T_4 \cup T_6$.*

Proof. For a tree T in $T_2 \cup T_4 \cup T_6$ it is easily seen that ${}_t(T) = \frac{2}{3}dd(T)$. For the converse, suppose that T is a tree for which ${}_t(T) = \frac{2}{3}dd(T)$ and let D be a minimum total dominating set. Since the vertices of D contribute at most ${}_t(T)$ distinct degrees, the vertices of $V(T) - D$ must contribute at least $\frac{{}_t(T)}{2}$ distinct degrees, which implies that there must be a vertex of degree at least $\frac{{}_t(T)}{2}$ in $V - D$. Now together with Lemma 10 we have that there is a vertex v of degree $\frac{{}_t(T)}{2}$. Next by Lemma 12, $\frac{3}{2}\ {}_t(T) = dd(T) \leq {}_t(T) + 3$, and it follows that ${}_t(T)$ must be even and at most 6. We proceed by considering three cases with the observation that since v is of degree $\frac{{}_t(T)}{2}$, D

must induce the union of P_2s (paths on two vertices).

Case 1: Suppose $_t(T) = 2$ and $dd = 3$. Then D induces a P_2. Clearly, every vertex in $V(T) - D$ is adjacent to at exactly one endpoint of P_2 and must be a leaf in T. Thus, T is a binary star. Now, since $dd = 3$, and every vertex in $V(T) - D$ is a leaf, the two vertices in D must be of distinct degree, and thus $T \in T_2$.

Case 2: Suppose $_t(T) = 4$ and $dd = 6$. Then there is a vertex v in $V(T) - D$ that is of degree 2. Now, either v has both neighbors in D or one in D and one in $V(T) - D$. Suppose v has both neighbors in D. Then since T is a tree, no other vertex of $V(T) - D$ has more than one neighbor in D, that is they are leaves. So the vertices in $V(T) - D$ contribute two distinct degrees, namely degrees 1 and 2. Since $dd = 6$, each of the 4 vertices in D must be of degree at least 3 and distinct from one another. Thus, T is a graph in T_4. On the other hand, if v has one neighbor, say w, in D and another, call it u, in $V(T) - D$, then u must also have a neighbor in D. Now clearly w and u cannot be incident with a common component in D, but this implies that the remaining vertices of $V(T) - D$ are adjacent to exactly one vertex in D. Thus the vertices in $V(T) - D$ contribute two to the number of distinct degrees, namely degrees 1 and 2. Moreover, the 4 vertices in D must be degree at least 3 and distinct from one another. Thus again it follows that T is in T_4.

Case 3: Suppose $_t(T) = 6$ and $dd = 9$. Then there is a vertex v in $V(T) - D$ that is of degree 3. Now, either v has one neighbor in D or two in D, but not all three in D (otherwise $dd \neq 9$). Suppose that v has one neigbor, call it w, in D. Call the two neighbors of v that are in $V(T) - D$ a and b. Then a and b must be adjacent to some vertex in D, but not incident with the component containing w, nor incident to a common component of D. But this implies that the other vertices of $V(T) - D$ are adjacent to only one vertex each in D. Thus the vertices in $V(T) - D$ contribute three to the number of distinct degrees, namely degrees 1, 2 and 3. Moreover, the 6 vertices in D must be degree at least 4 and distinct from one another. Thus T is in T_6. Finally, suppose that v has two neighbors in D. Then each of these two must be part of distinct components of D and so the third neighbor of v (the one in $V(T) - D$)) must be adjacent to the third component of D. Now, clearly the remaining vertices of $V(T) - D$ are adjacent to exactly one vertex in D. Thus the vertices in $V(T) - D$ contribute three to the number of distinct degrees, namely degrees 1, 2 and 3. Moreover, the 6 vertices in D must be degree at least 4 and distinct from one another. Thus again it follows that T is in T_6.

□

3 Some Open Conjectures

Graffiti.pc proposed lower bounds on the total domination number of a tree that involve the number of components of the subgraph induced by the maximum degree vertices. Before discussing one of those we prove a related simple lower bound.

Proposition 14. *Let T be a non-trivial tree, M the set of vertices of maximum degree in T and c_Δ be the number of components of the subgraph induced by M. Then*

$$ {}_t(T) \geq c_\Delta + 1. $$

Proof. We assume that T has $\Delta \geq 3$ and at least 2 components induced by the maximum degree vertices otherwise the relation follows trivially. Let $\Gamma(T)$ be the set of components induced by the maximum degree vertices of T, and let $c_\Delta = |\Gamma(T)|$. Consider a component $C_i \in \Gamma(T)$, and let x_i be a representative of C_i. Observe that every x_i has Δ neighbors that do not induce a maximum degree component of their own, either they are not of maximum degree or they are in C_i. Suppose that the number of vertices incident to more than one C_i is k. Then there are at least $\Delta c_\Delta - k$ vertices that will not contribute to c_Δ, that is $c_\Delta \leq n - (\Delta c_\Delta - k)$. The latter is equivalent to $(\Delta + 1)c_\Delta \leq n + k$. Now since T is a tree, $k \leq c_\Delta - 1$, and we see that

$$ c_\Delta \leq \frac{n-1}{\Delta}. $$

Now, it is easily seen that the claim follows from the fact that $\frac{n}{\Delta} \leq {}_t(T)$. □

From Theorem 11 and the above proposition, it is easily seen that ${}_t(T) \geq \frac{1}{2}c_\Delta + \frac{1}{3}dd(T)$. Graffiti.pc proposed the following improvement over the latter observation.

Conjecture 1. *[Graffiti.pc #379] Let T be a non-trivial tree, M the set of vertices of maximum degree in T and c_Δ be the number of components of the subgraph induced by M. Then*

$$ {}_t(T) \geq c_\Delta + \frac{dd(T)}{3}. $$

The next two conjectures of Graffiti.pc also suggest improvement for trees over known results for all connected graphs. In [4], it is proven that for any connected graph G, ${}_t(G) \geq rad(G)$, and that ${}_t(G) \geq (diam(G) + 1)/2$.

Conjecture 2. *[Graffiti.pc #349] Let T be a non-trivial tree, S the set of vertices of degree 2 in T and c be the number of components of the subgraph induced by $N(S) \cup S$. Then*

$$ _t(T) \geq rad(T) + c - 1.$$

Conjecture 3. *[Graffiti.pc #350] Let T be a non-trivial tree, S the set of vertices of degree 2 in T and c be the number of components of the subgraph induced by $N(S) \cup S$. Then*

$$ _t(T) \geq \frac{diam(T) + c}{2}.$$

In [1], Blidia, Chellali and Maffray present a new upper bound for the domination number of a graph G, which we denote by (G), and is defined as the minimum cardinality of a set S such that every vertex not in the set has a neighbor in the set. Let $_v(G)$ be the maximum size of a matching in the subgraph induced by the non-neighbors of v and put $\Delta'(G) = max\{d(v) + \ _v(G) | v \in V(G)\}$. Specifically, they proved that for any graph $G = (V, E)$, $(G) \leq |V(G)| - \Delta'(G)$. Recently, DeLaViña introduced the graph invariant $\Delta'(G)$ to Graffiti.pc, which it used to conjecture the following.

Conjecture 4. *[Graffiti.pc #370] Let T be a non-trivial tree. Then*

$$ _t(T) \geq \frac{2\Delta'(T)}{\Delta(T)}.$$

We end this paper with a couple of additional open conjectures and one refuted conjecture. For a tree T with p_2 the order of a largest path in the subgraph induced by the degree two vertices of T, and $S(T)$ the set of support vertices of T, it is easy to see that $_t(T) \geq \frac{p_2}{2} + |S(T)| - 2$. Graffiti.pc proposed the following slight improvement. Although it may also be easily resolved, we note that few improvements for the obvious $_t(T) \geq |S(T)|$ are known.

Conjecture 5. *[Graffiti.pc #347] Let T be a non-trivial tree, p_2 the order of a largest path in the subgraph induced by the degree two vertices, and $S(T)$ the set of support vertices of T . Then*

$$ _t(T) \geq \frac{p_2}{2} + |S(T)| - 1.$$

Conjecture 6. *[Graffiti.pc #367] Let T be a non-trivial tree in which the most frequently occuring degree is degree 2. Then*

$$ _t(T) \geq \frac{4}{3} dd(T).$$

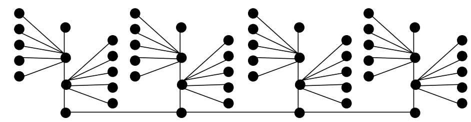

Figure 5: Counter-example to Graffiti.pc's 368

Let $d = (d_1, d_2, d_3, ..., d_n)$ be a non-decreasing sequence of non-negative integers, and $d' = (d_2 - 1, d_3 - 1, ..., d_{d_1+1} - 1, d_{d_1+2}, ..., d_n)$ its *derivative*. A sequence is said to be *graphic* if it is the degree sequence for some graph. A result of Havil [9] and Hakimi [8] states that a sequence d graphic if and only if d' is graphic. Let G be a graph and $d(G)$ its degree sequence in non-decreasing order. It is also known that if one repeats the derivative on the degree sequence of a graph, then after some steps the resulting sequence will be a zero-sequence. Incidentally, the number of zeros of the resulting sequence is called the *residue of the graph*, a term coined by Fajtlowicz; moreover, his program Graffiti conjectured that the independence number of a graph is at least its residue, which was proven in [7]. Now for a graph G, let us call the entire process of repeating the derivative on $d(G)$, the Havil-Hakimi process, and let k be the smallest integer such that in the Havil-Hakimi process the k^{th} step introduces a zero in the derivative. DeLaViña introduced the graph invariant to Graffiti.pc, which it used to conjecture the following.

Conjecture 7. *[Graffiti.pc #368] Let T be a tree on $n > 2$ vertices. Then*

$$_t(T) \geq 1 + k,$$

where the k^{th} step of the Havil-Hakimi process introduces a zero.

The graph in Figure 5 is a counter-example to this conjecture with total domination number 8 and $k = 8$. It is easily seen that this counter-example belongs to a large family of graphs that serve as counter-examples, but we are interested in knowing if a smaller counter-example exists.

References

[1] M. Blidia, M. Chellali, F. Maffray, Extremal graphs for a new upper bound on domination parameters in graphs, *Discrete Mathematics*, **306** (2006), 2314-2326.

[2] M. Chellali and T. W. Haynes, A note on the total domination number of a tree. *J. Combin. Math. Combin. Comput.* **58** (2006), 189-193.

[3] E. J. Cockayne, R. M. Dawes, and S. T. Hedetniemi, Total domination in Graphs. *Networks* **10** (1980), 211-219.

[4] E. DeLaViña, Q. Liu, R. Pepper, B. Waller and D. B. West, On some conjectures of Graffiti.pc on total domination, *Proceedings of the Thirty-Seventh Southeastern International Conference on Combinatorics, Graph Theory and Computing, Congressus Numerantium*, **185** (2007), 81-95.

[5] E. DeLaViña, Some history of the development of Graffiti, *Graphs and Discovery DIMACS: Series in Discrete Mathematics and Theoretical Computer Science*, **69** (2005), 81-118.

[6] E. DeLaViña, Written on the Wall II, Web address: http://cms.dt.uh.edu/faculty/delavinae/research/wowII

[7] O. Favaron, M. Maheo and J-F. Sacle, On the Residue of a Graph *Journal of Graph Theory*, **15** (1991), 39-64.

[8] S. Hakimi, On the Realizability of a Set of Integers as Degrees of the Vertices of a Graph, *SIAM J. Appl. Math.*, **10** (1962), 496-506.

[9] V. Havel, A Remark on the Existence of Finite Graphs [Czech], *Casopis Pest. Mat.*, **80** (1955), 477-480.

[10] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, *Fundamentals of Domination in Graphs,* Mercel Dekker, New York (1998).

[11] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, *Domination in Graphs: Advanced Topics,* Mercel Dekker, New York (1998).

[12] M. A. Henning, A survey of selected recent results on total domination in graphs, *Discrete Mathematics*, **309** (2009), 32-63.

[13] R.C. Laskar, J. Pfaff, S.M. Hedetniemi and S.T. Hedetniemi, On the algorithmic complexity of total domination, *SIAM J. Algebraic Discrete Methods*, **5** (1984), 420-425.

[14] J. Pfaff, R.C. Laskar, S.T. Hedetniemi, NP-completeness of total and connected domination and irredundance for bipartite graphs, Technical Report 428, Clemson University, Dept. Math. Sciences, (1983).